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Exact solutions of the invariant density for piecewise linear approximation to cubic maps

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Abstract. A two-parameter piecewise linear approximation to cubic maps has been proposed and analytically investigated. The notion of (n, k) sequences has been introduced to find the exact solutions of the invariant density of our model, via the Frobenius-Perron operator. For a one-parameter version of the present map we obtain the exact invariant density and the corresponding parameter equation. Our results can be regarded as a generalisation to the consequences obtained by Derrida *et al* for the tent map.

1. Introduction

Non-linear dissipative systems exhibiting a large number of diverse and complicated types of behaviours have been extensively studied by using the iteration functions with one critical point [1-6]. A rich structure of subharmonic bifurcations, periodic doublings, universal functions, noisy bands and chaos has been found [2]. The dynamical properties of the cubic maps with two extrema have attracted much attention in recent years [7-10]. The numerical results as well as the theoretical analysis on such cubic maps show up various interesting features which are not found in the unimodal maps.

In this paper we study a piecewise linear approximation to cubic maps

$$f(x)=\begin{cases} ax+b & 0\leqslant x\leqslant x_1 \\ -ax+2-b & x_1\leqslant x\leqslant x_2 \\ ax+b-2 & x_2\leqslant x\leqslant 1 \end{cases} \tag{1.1}$$

where $x_1=(1-b)/a$, $x_2=(2-b)/a$, x_1 and x_2 are the critical points of f, and a, b are two parameters satisfying $2<a\leqslant 3$ and $0<b<3-a$. The main interest of the presentation focus on finding the exact solutions of the operator equation

$$HP(x)=P(x) \tag{1.2}$$

where H is the Frobenius-Perron operator corresponding to map ((1.1), and $P(x)$ is an invariant density. Before proceding we first discuss the uniqueness of solutions of this operator equation.

For a class of piecewise continuous, piecewise c^1 transformations on the interval $J\subset R$ with finitely many discontinuities n, Li and Yorke [11, 12] have proved that there exist at most n invariant measures. From the theorem 1 of [11] Grossmann and Thomae [13] have formulated the following criterion of uniqueness for the invariant density.

Corollary. Let $\tau:[0,1]\to[0,1]$ be a piecewise c^2 function such that

$$\inf_y |d\tau(y)/dy| > 1.$$

Let the set $J := \{y_0, y_1, \ldots, y_n\}$ of points, where $d\tau(y)/dy$ does not exist, be finite, and

$$U_\varepsilon(y_j) := (y_j - \varepsilon, y_j + \varepsilon) \cap (0, 1) \qquad \varepsilon > 0$$

be the ε neighbourhood of any point of discontinuity y_j. The invariant density $P(x)$ is unique if there are positive integers n_1, n_2 for each pair (y_i, y_j) such that for arbitrarily small $\varepsilon > 0$

$$m(f^{n_1}[U_\varepsilon(y_i)] \cap f^{n_2}[U_\varepsilon(y_j)]) \neq 0$$

where $m(I)$ denotes the size of the interval I (Lebesque measure). Thus the uniqueness of the invariant density for map (1.1) follows immediately from the corollary stated above since the chaotic attractor of f is the whole interval [0, 1] in our cases.

In the remaining sections we calculate the invariant density through equation (1.2) and find the exact solutions of this equation for the (n, k) sequences of map (1.1) (the definition of (n, k) sequences will appear in § 2) for some special values of parameter b. We also derive the corresponding characteristic equations for parameter a, which are found to be very useful in understanding the symbolic dynamics for map (1.1) and perhaps for the general cubic maps. Finally we make some remarks on the convergence of initial densities to the invariant density.

2. A piecewise linear approximation to cubic maps with one parameter

When $b = (3 - a)$, equation (1.1) becomes

$$f(x) = \begin{cases} ax + (3-a)/2 & 0 < x < x_1 \\ -ax + (a+1)/2 & x_1 < x < x_2 \\ ax - (a+1)/2 & x_2 < x < 1 \end{cases} \tag{2.1}$$

where a is the control parameter which varies between 2 and 3. $x_1 = (a-1)/2a$ and $x_2 = (a+1)/a$ are the critical points of map (2.1). In view of $f(x_1) = 1$ and $f(x_2) = 0$, we hereafter take simply 1 and 0 as the critical points for the convenience of manipulation. The map (2.1) is plotted in figure 1.

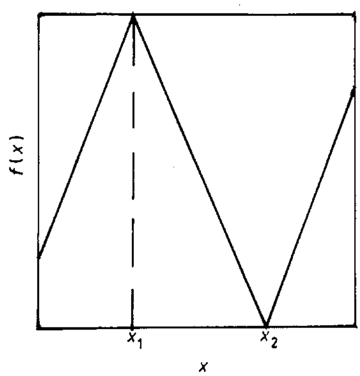

Figure 1. The symmetric piecewise linear approximation to cubic map. x_1 and x_2 are the critical points.

In order to find the invariant density we introduce the Frobenius-Perron operator

$$HP(x)=\sum_i |[(\mathrm{d}/\mathrm{d}x)f_i^{-1}(x)]|P(f_i^{-1}(x))\chi[f_i^{-1}(x)] \tag{2.2}$$

where $f_i^{-1}(x)$ are different branches of the inverse of $f(x)$ and $\chi[f_i^{-1}(x)]$ is the indicator function of the branch i. Thus the invariant density $P(x)$ is determined by the following equation

$$HP(x)=P(x). \tag{2.3}$$

As far as equation (2.1) is concerned, the operator H is

$$\begin{aligned} HP(x)=[&P((2x+a+1)/2a)\theta(f(1)-x)+P((a+1-2x)/2a)\\ &+P((2x-3+a)/2a)\theta(x-f(0))]/a. \end{aligned} \tag{2.4}$$

Since we are only interested in finding the exact solutions of equation (2.3), it is reasonable to investigate the conditions under which the invariant density can be derived exactly. A careful examination of the geometrical structure of map (2.1) reveals that the exact solution may be obtained for those parameter values determined by

$$f^{n+k}(a;x_i)=f^n(a;x_i) \qquad i=1,2 \tag{2.5}$$

where $x_1(=1)$ and $x_2(=0)$ are regarded as the critical points of (2.1). When $n=0$ one has the critical periodic orbits, otherwise the eventually critical periodic orbits, i.e. the (n,k) sequences which are defined as

$$\{f^i(a;0),\ i=1,2,\ldots,n+k-1;f^{n+k}(a;0)=f^n(a;0)\} \tag{2.6}$$

and

$$\{f^j(a;1), j=1,2,\ldots,n+k-1;f^{n+k}(a;1)=f^a(a;1)\}. \tag{2.7}$$

Equations (2.6) and (2.7) imply that the trajectories fall into the cycle of period k after n iterates, starting from $x_0=0$ or $x_0=1$, respectively. It is obvious that the properties of the (n,k) sequences may be described successfully in the sense of Metropolis *et al* (MSS) [4]. In the tent map the MSS sequences are the special cases of its (n,k) sequences with $n=0$, while in the cubic map the (n,k) sequences may be described by the generalised notation of the MSS sequences. With a more detailed study on the system (2.1) one may find that there exist two kinds of (n,k) sequences; one being the coexisting (n,k) sequences as defined by equations (2.6) and (2.7), and the other being the so-called isolated sequences. The isolated trajectory passes through two critical points of map (2.1) at the same time while two of the coexisting orbits go through their own respective critical points. It is easy to see that those (n,k) sequences generate finite partitions on the interval [0, 1], and make it possible to find the exact solutions of equation (2.3). In what follows we compute the invariant density only for the coexisting (n,k) sequences. The calculation for the isolated (n,k) sequences follows the same line discussed here. Now we assume that

$$P(x)=a_0+\sum_i^{n+k-1} a_i\theta(x-f^i(0))+\sum_j^{n+k-1} b_j\theta(x-f^i(1)). \tag{2.8}$$

By applying operator H on $\theta(x-f^i(0))$, one obtains

$$H\theta(x-f^i(0))=[\alpha_1\theta(x-f^{i+1}(0))+2\beta_i-\theta(x-f(1))]/a \tag{2.9}$$

with α_i and β_i defined by

$$\alpha_0 = 1$$
$$\alpha_i = \begin{cases} 1 & 0 < f^i(0) < x_1 \text{ or } x_2 < f^i(0) < 1 \\ -1 & x_1 < f^i(0) < x_2 \end{cases} \tag{2.10}$$

and

$$\beta_0 = 1$$
$$\beta_i = \begin{cases} 1 & 0 < f^i(0) < x_2 \\ 0 & x_2 < f^i(0) < 1. \end{cases} \tag{2.11}$$

Similarly one has

$$H\theta(x - f^j(1)) = [\alpha'_j \theta(x - f^{j+1}(1)) + 2\beta'_j - \theta(x - f(1))]/a \tag{2.12}$$

where α'_j and β'_j are defined by

$$\alpha'_0 = 1$$
$$\alpha'_j = \begin{cases} 1 & 0 < f^j(1) < x_1 \text{ or } x_2 < f^j(1) < 1 \\ -1 & x_1 < f^j(1) < x_2 \end{cases} \tag{2.13}$$

and

$$\beta'_0 = 0$$
$$\beta'_j = \begin{cases} 1 & 0 < f^j(1) < x_2 \\ 0 & x_2 < f^j(1) < 1. \end{cases} \tag{2.14}$$

Since

$$Ha_0 = a_0[2 + \theta(x - f(0)) - \theta(x - f(1))]/a \tag{2.15}$$

we obtain

$$\begin{aligned} HP(x) = {} & a_0[2 + \theta(x - f(0)) - \theta(x - f(1))]/a \\ & + \sum_{i=1}^{n+k-1} a_i[\alpha_i \theta(x - f^{i+1}(0)) + 2\beta_i - \theta(x - f(1))]/a \\ & + \sum_{j=1}^{n+k-1} b_j[\alpha'_j \theta(x - f^{j+1}(1)) + 2\beta'_j - \theta(x - f(1))]/a. \end{aligned} \tag{2.16}$$

Substituting (2.8) and (2.16) into (2.3) and comparing the coefficients of $\theta(x - f^i(0))$ and $\theta(x - f^i(1))$, we finally obtain the characteristic equation

$$\begin{aligned} a = 2\Bigg(& 1 + (1 - \delta_{n,0})(1 - \delta_{n,1}) \sum_{i=1}^{n-1} (\alpha_1 \alpha_2 \ldots \alpha_{i-1} \beta_i)/a^i + (\alpha_1 \alpha_2 \ldots \alpha_{k-1} \delta_{n,0})/2a^{k-1} \\ & + (1 - \delta_{k,1}) \sum_{i=1}^{k-1} (\alpha_n \ldots \alpha_{n+i-1} \beta_{n+i}) D(n, k; \alpha)/a^i \\ & + (1 - \delta_{n,0})(\alpha_1 \alpha_2 \ldots a_{n-1} \beta_a) D(n, k; \alpha)/a^n \\ & - (1 - \delta_{n,0})(1 - \delta_{n,1}) \sum_{i=1}^{n-1} (\alpha'_1 \alpha'_2 \ldots \alpha'_{i-1} \beta'_i) W(n, k; \alpha, \alpha')/a^i \\ & - (1 - \delta_{k,1}) \sum_{i=1}^{k-1} \alpha'_n \ldots \alpha'_{n+i-1} \beta'_{n+i}) D(n, k; \alpha') W(n, k; \alpha, \alpha')/a^i \\ & - (1 - \delta_{n,0}) \alpha'_1 \alpha'_2 \ldots \alpha'_{n-1} \beta'_n) D(n, k; \alpha') W(n, k; \alpha, \alpha')/a^n \Bigg) \end{aligned} \tag{2.17}$$

and the corresponding invariant density

$$P(x)=a_0[1+(1-\delta_{n,0})(1-\delta_{n,1})\sum_{i=1}^{n-1}(\alpha_1\alpha_2\ldots\alpha_{i-1})\theta(x-f^i(0))/a^n-(1-\delta_{n,0})(1-\delta_{n,1})$$
$$\times\sum_{i=1}^{n-1}(\alpha'_1\alpha'_2\ldots\alpha'_{i-1})W(n,k;\alpha,\alpha')\theta(x-f^i(1))/a^i]$$
$$+(a_0/a^n)(1-\delta_{n,0})(\alpha_1\alpha_2\ldots\alpha_{n-1})\theta(x-f^n(0))$$
$$-(a_0/a^n)(1-\delta_{n,0})(\alpha'_1\alpha'_2\ldots\alpha'_{n-1})W(n,k;\alpha,\alpha')\theta(x-f^n(1))$$
$$+a_0(1-\delta_{k,1})\sum_{i=1}^{k-1}[\alpha_n\alpha_{n+1}\ldots\alpha_{n+i-1})D(n,k;\alpha)\theta(x-f^{n+i}(0))$$
$$-(\alpha'_n\alpha'_{n+1}\ldots\alpha'_{n+i-1})D(n,k;\alpha)W(n,k;\alpha,\alpha')\theta(x-f^{n+i}(1))]/a^i \tag{2.18}$$

where

$$D(n,k;\alpha)=(\alpha_1\alpha_2\ldots\alpha_{n-1})/(a^n-a^{n-k}T(n,k)) \tag{2.19}$$
$$T(n,k)=\alpha_n\alpha_{n+1}\ldots\alpha_{n+k-1} \tag{2.20}$$
$$W(n,k;\alpha,\alpha')=\Bigg(1+(1-\delta_{n,0})(1-\delta_{n,1})\sum_{i=1}^{n-1}(\alpha_1\alpha_2\ldots\alpha_{i-1})/a^i$$
$$+(1-\delta_{k,1})\sum_{i=1}^{k-1}(\alpha_n\alpha_{n+1}\ldots\alpha_{n+i-1})D(n,k;\alpha)$$
$$+(1-\delta_{n,0})(\alpha_1\alpha_2\ldots\alpha_{n-1})/a^n\Bigg)$$
$$\times\Bigg((\delta_{n,1}\alpha'_1\alpha'_2\ldots\alpha'_k)/a^k+1+(1-\delta_{n,0})(1-\delta_{n,1})\sum_{i=1}^{n-1}(\alpha'_1\alpha'_2\ldots\alpha'_{i-1})/a^i$$
$$+(1-\delta_{k,1})\sum_{i=1}^{k-1}(\alpha'_n\ldots\alpha'_{n+i-1})D(n,k;\alpha')/a^{i-1}$$
$$+(1-\delta_{n,0})(\alpha'_1\alpha'_2\ldots\alpha'_{n-1})/a^n\Bigg)^{-1}. \tag{2.21}$$

When $n=0$ we return to the case of the critical periodic orbits with the parameter equation and invariant density being given by

$$a=2+2(1-\delta_{k,1})\sum_{i=1}^{k-1}\alpha_1\alpha_2\ldots\alpha_{i-1}\beta_i/a^i+(\alpha_1\alpha_2\ldots\alpha_{k-1})/a^k$$
$$-2\sum_{j=1}^{k-1}(\alpha'_1\alpha'_2\ldots\alpha'_{j-1}\beta'_j)W(0,k;\alpha,\alpha')/a^j \tag{2.22}$$

and

$$P(x)=a_0\Bigg(1+\sum_{i=1}^{k-1}(\alpha_1\alpha_2\ldots\alpha_{i-1})\theta(x-f^i(0))/a^i$$
$$-\sum_{i=1}^{k=1}(\alpha'_1\alpha'_2\ldots\alpha'_{j-1})W(0,k;\alpha,\alpha')\theta(x-f^j(1))/a^j\Bigg). \tag{2.23}$$

The special examples of these results are illustrated in figure 2.

Equations (2.17) and (2.18) provide much information about the dynamical and statistical properties of map (2.1). For example, from (2.18) one can study the structure

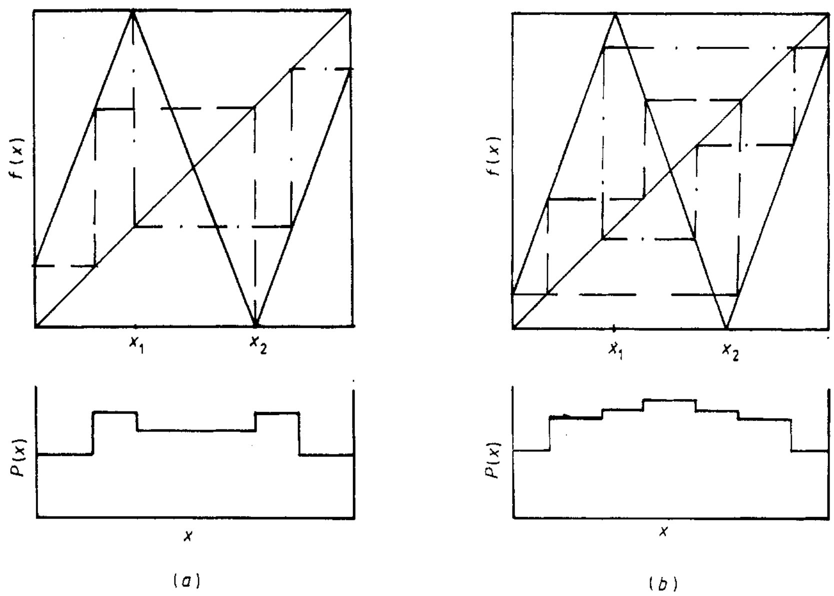

Figure 2. (*a*) A pair of coexisting periodic critical orbits with $n=0$ and $k=3$, labelled by $x_2 \to L^2$ and $x_1 \to R^2$, respectively. (*b*) A pair of coexisting eventually periodic critical orbits with $n=1$ and $k=3$, labelled by $x_2 \to L$ (LMR) and $x_1 \to R$ (RML), respectively.

and the order of the (n, k) sequences via the method developed by Derrida *et al* for the tent map [14]. To uniquely determine the patterns of the (n, k) sequences in the cubic map, one has to divide the iterative map into its three monotonically increasing or decreasing regimes:

$$L: \quad 0 < x < x_1$$
$$M: \quad x_1 < x < x_2$$
$$R: \quad x_2 < x < 1.$$

By means of this notion one may assign each cycle an itinerary representing the order in which the point of a definite (n, k) sequence visits the different regimes of the map (2.1). This itinerary generalises the Metropolis symbol which is solely formed by two letters R and L.

3. General remarks on the properties of map (1.1)

We now study the following map:

$$f(a, b; x) = \begin{cases} ax + b & 0 < x < x_1 \\ -ax - b + 2 & x_1 < x < x_2 \\ ax + b - 2 & x_2 < x < 1 \end{cases} \tag{3.1}$$

where $x_1 (= (1-b)/a)$ and $x_2 (= (2-b)/a)$ are the critical points and a and b are the control parameters satisfying $2 < a < 3$ and $0 < b < 3 - a$.

In this two-parameter, piecewise linear approximation to the cubic map one can discover a richer structure not observed in the one-parameter cubic maps [15–17]. With parameters a and b changed, the system (3.1) behaves in a very complicated manner, which makes the analytical discussion on its invariant density more difficult. Thus in the following we restrict ourselves to the brief analysis on map (3.1) for three special values of b.

Case A: $b=(3-a)/2$. When the parameter b is chosen to be $(3-a)/2$ one returns to the situation discussed in § 2. It is easy to see that the invariant density is symmetric since the corresponding map is itself symmetric as well.

Case B: $b=3-a$. In this case the map (3.1) turns out to be

$$f(a,x)=\begin{cases} ax-a+3 & 0<x<x_1 \\ -ax+a-1 & x_1<x<x_2 \\ ax-a+1 & x_2<x<1. \end{cases} \tag{3.2}$$

In this peculiar map there is only one critical point $(x=(a-1)/a)$, the other critical point $(x_1=(a-2)/a)$ iterates to the fixed point $(x_f=1)$ in one step. Because of this feature there exist no coexisting (n,k) sequences in (3.2), and its invariant density exhibits no symmetry (see figure 3).

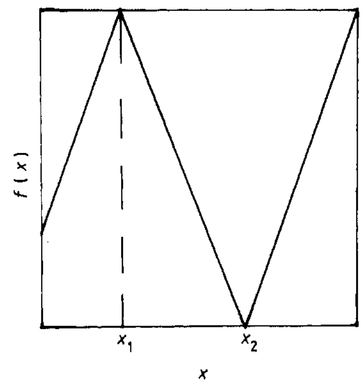

Figure 3. The piecewise linear approximation to the cubic map with one critical point.

According to the procedure described in § 2 one may easily find the characteristic equation and invariant density of the (n,k) sequences for (3.2). They are

$$\begin{aligned} a=2&+2(1-\delta_{n,0})(1-\delta_{n,1})\sum_{i=1}^{n-1}(\alpha_1\alpha_2\ldots\alpha_{i-1}\beta_i)/a^i \\ &+(1-\delta_{k,1})\sum_{j=1}^{k-1}(\alpha_1\alpha_2\ldots\alpha_{n+j-1}\beta_{n+j})/(a^{n+j}-a^{n-k+j}T(n,k)) \\ &+2(1-\delta_{n,0})(\alpha_1\alpha_2\ldots\alpha_{n-1})/(a^n-a^{n-k}T(n,k)) \\ &+(\alpha_1\alpha_2\ldots\alpha_{k-1})\delta_{n,0}/a^{k-1} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} P(x)=a_0[1&+(1-\delta_{n,0})(1-\delta_{n,1})\sum_{i=1}^{n-1}(\alpha_1\alpha_2\ldots\alpha_{i-1})\theta(x-f^i(0))/a^i \\ &+(1-\delta_{n,0})(\alpha_1\alpha_2\ldots\alpha_{n-1})\theta(x-f^n(0))/(a^a-a^{n-k}T(n,k)) \\ &+(1-\delta_{k,1})\sum_{j=1}^{k-1}(\alpha_1\alpha_2\ldots\alpha_{n+j-1})\theta(x-f^{n+j}(0))/(a^{n+j}-a^{n+j-k}T(n,k)). \end{aligned} \tag{3.4}$$

So far one finds that the parameter a may be obtained through an algebraic equation similar to the so-called a autoexpansion in the case of one-dimensional unimodal maps. It is easily noticed that not all possible sequences from $\{\alpha_i\}$ and $\{\beta_i\}$ contribute a solution of the parameter equation for a ranging between 2 and 3. Hence a necessary condition for admissible sequences can certainly be extracted by analysing the parameter equation. From numerical calculation we suggest that for the admissible (n, k) sequences the following condition:

$$\pm(a_n, a_{n+1}, \ldots) < (a_0, a_1, \ldots)$$

should be satisfied [14].

Case C: $0 < b < (3-a)/2$ and $(3-a)/2 < b < (3-a)$. In this case one has a two-parameter non-symmetric piecewise linear approximation to the cubic map, which exhibits various kinds of orbits such as the isolated (n, k) sequences, coexisting (n, k) sequences with different n and k, etc. However, provided that there exists some kind of (n, k) sequences for certain values of a and b, one can compute the corresponding invariant density via equation (1.2) by assuming that

$$P(x) = a_0 + \sum_{i=1}^{n1+k1-1} a_i\theta(x - f^i(0)) + \sum_{j=1}^{n2+k2-1} b_j\theta(x - f^j(1)) \qquad (3.5)$$

for the first (n, k) sequences

$$\{f^i(0), i = 1, 2, \ldots, n1 + k1 - 1; f^{n1+k1}(0) = f^{n1}(0)\} \qquad (3.6)$$

starting with $x_2 = 0$, and the second (n, k) sequences

$$\{f^j(1), j = 1, 2, \ldots, n2 + k2 - 1; f^{n2+k2}(1) = f^{n2}(1)\} \qquad (3.7)$$

beginning from $x_1 = 1$. In principle, the analytical expression of $P(x)$ can be obtained via the Frobenius-Perron operator though the calculation is in fact very tedious. Therefore, instead of finding the general results for arbitrary (n, k) sequences we give only an example of $n1 = n2 = 0$, $k_1 = 4$ and $k_2 = 2$; see figure 4.

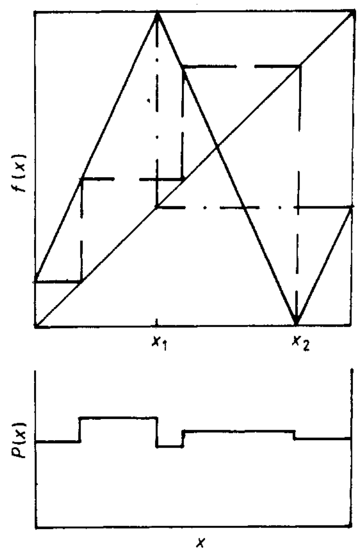

Figure 4. Map (3.1) and its invariant density at $a = 2.2393$ and $b = 0.1433$. The two distinct orbits are denoted $x_1 \to R$ and $x_2 \to L^2M$, respectively.

In previous sections we have presented a useful model for cubic maps, which can be investigated analytically. The notion of the (n, k) sequences has been introduced to find the exact solution of the invariant density and other features of that model. A detailed analysis on the properties of the (n, k) sequences has been conducted, which shows that finite partitions on the chaotic region of one-dimensional maps can be generated by the (n, k) sequences. These finite partitions enable us to evaluate the Stefan matrix [14] of the system considered. Notice that all the (n, k) sequences are unstable no matter what kind of map is considered. This implies that the (n, k) sequences exist only for those parameter values at which there are no stable orbits of the system. Thus one can deduce that the (n, k) sequences may be a particularly effective approach to understanding the dynamical properties of the chaotic systems.

Nevertheless, there is still a problem about the convergence to the invariant density for arbitrary densities. Losata *et al* [18] have shown that for a class of Markov operators acting on an arbitrary space $L'(X, \Sigma, \mu)$ with a σ-finite measure μ, the sequences $\{H^n P'\}$ with normalised P' converge to a compact set $\Omega = \{P_1, P_2, \ldots, P_r\}$. This implies that the Markov operator H is asymptotically periodic if $r > 1$ or asymptotically stable if $r = 1$. For the Frobenius-Perron operator H, which is induced by map (1.1), one may easily prove that the operator H is asymptotically stable in our case. First of all consider the case of $a = 3$ and $b = 0$. We may show that $H^n P'(x) \to P(x) = 1$ for all initial densities $P'(x)$.

Proof. By the definition of the Frobenius-Perron operator, one has

$$HP'(x) = \frac{1}{3}\left(\sum_{i=1}^{3} P'(f_i^{-1}(x))\right)$$

and therefore

$$H^n P'(x) = \frac{1}{3^n}\left(\sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\cdots\sum_{i_n=1}^{3} P'(f_{i_1}^{-1}(\ldots(f_{i_n}^{-1}(x))\ldots))\right).$$

Let $F_{i_1 i_2 \ldots i_n}(x) \equiv f_{i_1}^{-1}(\ldots(f_{i_n}^{-1}(x))\ldots)$ and σ be a permutation of integer sequences $\{i_1 i_2 \ldots i_n\}$. Then by setting $N = 3^n$ one has

$$H^n P'(x) = \frac{1}{N}\left(\sum_{m=1}^{N} P'(F_{\sigma^m(i)}(x))\right).$$

Since F maps the interval [0, 1] onto itself, we have

$$\lim_{n\to\infty} H^n P'(x) = \lim_{N\to\infty}\left[\frac{1}{N}\left(\sum_{m=1}^{N} P'(F_{\sigma^m(i)}(x))\right)\right] = P(x) = 1.$$

This conclusion is also true for those parameter values of a and b, determined by the (n, k) sequences of map (1.1), since the (n, k) sequences generate finite partitions on the interval [0, 1] which is the chaotic attractor of map (1.1). In this case the operator H can be represented by a finite-dimensional matrix $\mathcal{H}$. Since $\mathcal{H}$ is a stochastic indecomposable matrix, the largest eigenvalues are of modulus 1 and the largest real eigenvalue is 1 [13]. Hence one has

$$\mathcal{H}^n P'(x) \to P(x)$$

for any initial density $P'(x)$. Here $P(x)$ is the invariant density. Thus the exact solution given by (2.18) is indeed a unique solution of (1.2) and for all initial densities $P'(x)$,

$H^nP'(x)$ converges to this exact solution. It can be understood that, due to the finite partitions on the interval [0, 1] in our situation, equation (1.2) admits a unique solution and operator H is asymptotically stable, while the operator equation $HP(x) = P(x)$ will not, in general, have a unique solution and the operator H will be asymptotically periodic.

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